MMAT5360 200309

Nash bargaining model

(A, B) : A, B $m \times n$ matrices

Cooperative Games

Non-transferable utility (NTU)

Players make agreements on what strategies they use.

Sharing of payoffs are not allowed.

Def.

1. Joint strategies :

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{pmatrix} \quad m \times n \text{ matrix} \quad \begin{array}{l} ① \; p_{ij} \geq 0 \\ ② \; \sum p_{ij} = 1 \end{array}$$

Probability matrix

p_{ij} = probability that I uses i-th strategy and II uses j-th strategy

payoff : $(u(P), v(P)) = (\sum a_{ij} p_{ij}, \sum b_{ij} p_{ij})$

$$= \sum p_{ij} (a_{ij}, b_{ij})$$

In particular, if $\vec{p} \in \mathcal{P}^m$, $\vec{q} \in \mathcal{P}^n$

$$\underset{m \times n \text{ matrix}}{P = \vec{p}^T \vec{q}} = \begin{pmatrix} p_1 q_1 & p_1 q_2 \cdots & p_1 q_n \\ \vdots & \ddots & \vdots \\ p_m q_1 & \cdots & p_m q_n \end{pmatrix} \in \mathcal{P}^{m \times n}$$

I uses $\vec{p}$ and II uses $\vec{q}$ independently

Not all strategies are of this form e.g. $\begin{pmatrix} 0.4 & 0 \\ 0 & 0.6 \end{pmatrix}$

2. Cooperative region : $\mathcal{R} = \{ (u(P), v(P)) : P \in \mathcal{P}^{m \times n} \}$

$\mathcal{R} = \operatorname{conv}\{ (a_{ij}, b_{ij}) \}$ ① convex polygon in $\mathbb{R}^2$
② closed and bounded

3. Status quo point : payoff pair (μ, ν) associated to the solution when (A, B) is considered as

a non-cooperative game.

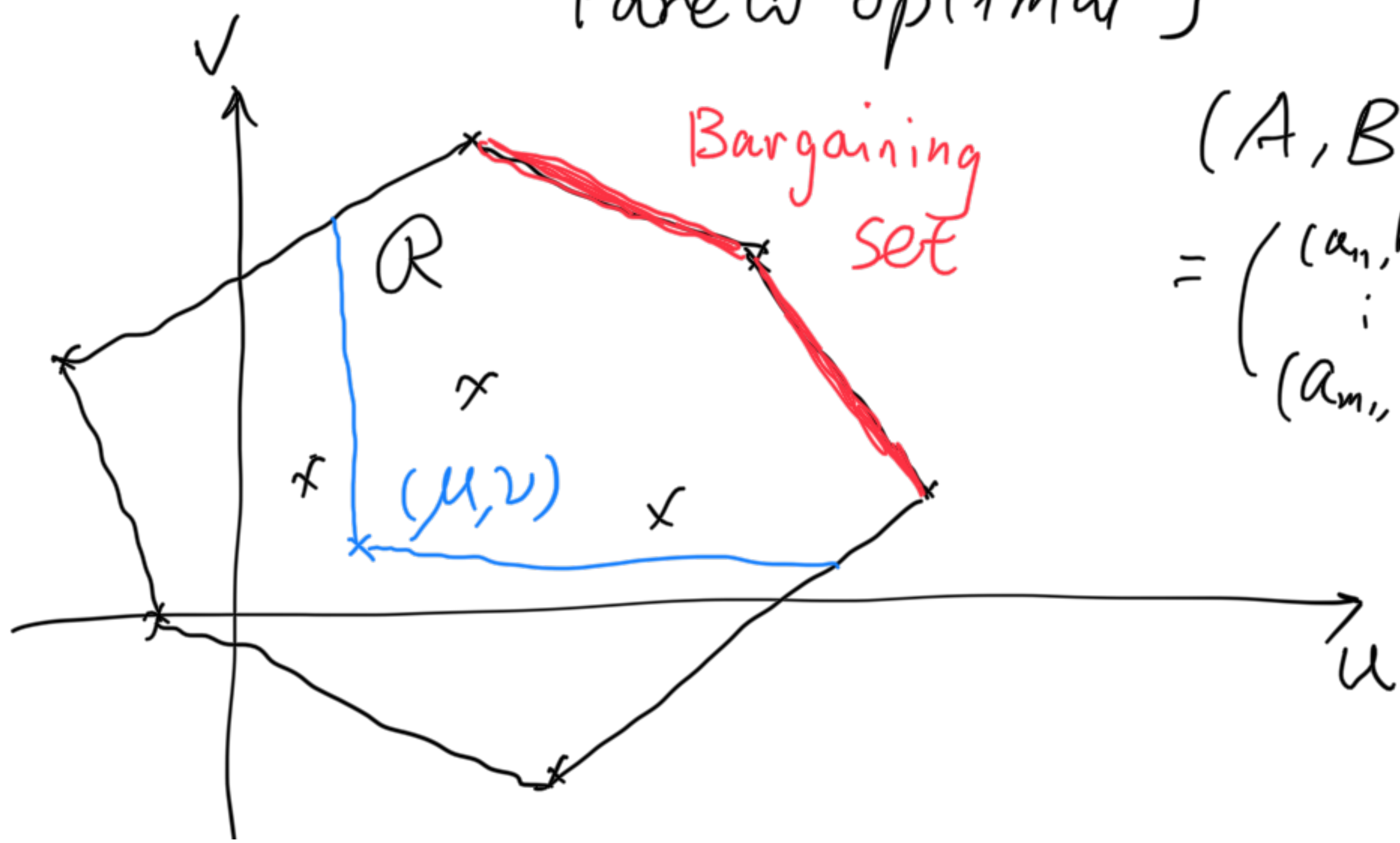
Unless otherwise specified, we take

$(\mu, \nu) = (v(A), v(B^T))$ security levels

4. We say that $(u,v) \in \mathcal{R}$ is <u>Paréto optimal</u> if $u' \geq u$, $v' \geq v$ and $(u',v') \in \mathcal{R}$ implies $(u',v') = (u,v)$

5. Bargaining set:

$$\{ (u,v) \in \mathcal{R} : (u,v) \geq (\mu, \nu) \text{ and } (u,v) \text{ is Paréto optimal} \}$$



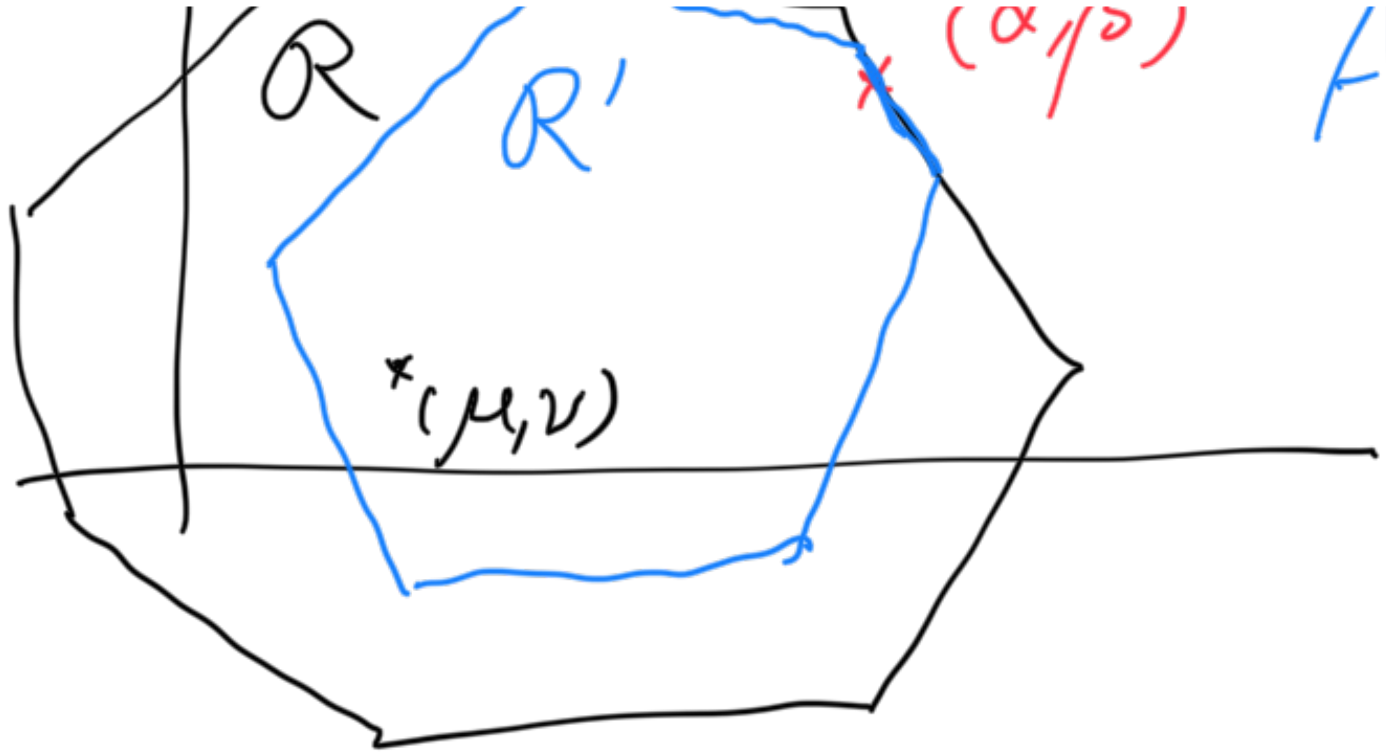
$$(A,B) = \begin{pmatrix} (a_{11}, b_{11}) & \cdots & (a_{1n}, b_{1n}) \\ \vdots & & \\ (a_{m1}, b_{m1}) & \cdots & (a_{mn}, b_{mn}) \end{pmatrix}$$

Nash's axiom for bargaining problem

An arbitration is a function $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$ defined for a closed and bounded convex set $\mathcal{R} \in \mathbb{R}^2$ and $(\mu,\nu) \in \mathcal{R}$ such that the following axioms are satisfied.

1. (Individual rationality) $(\alpha,\beta) \geq (\mu,\nu)$
2. (Paréto optimality) If $(u,v) \in \mathcal{R}$ and $(u,v) \geq (\alpha,\beta)$, then $(u,v) = (\alpha,\beta)$.
3. (Feasibility) $(\alpha,\beta) \in \mathcal{R}$
4. (Independence of irrelavent alternatives) If $\mathcal{R}' \subset \mathcal{R}$ and $(\mu,\nu), (\alpha,\beta) \in \mathcal{R}'$, then $$A(\mathcal{R}',(\mu,\nu)) = (\alpha,\beta)$$

(1),(2),(3) $\Rightarrow$ (α,β) lies on the bargaining set.



$A(\mathcal{R}',(\mu,\nu))=(\alpha,\beta)$

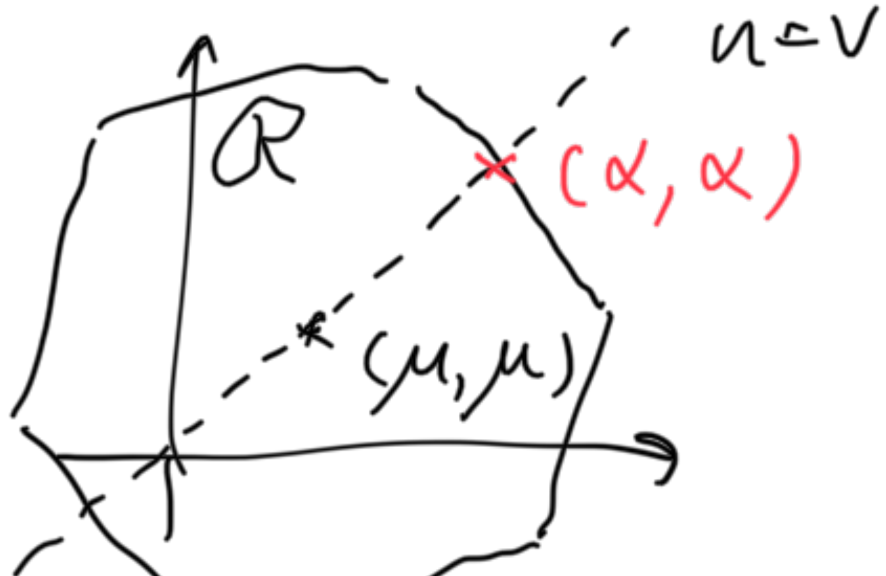
5. (Invariant under linear transformations)

$$\mathcal{R}'=\{(u',v') : u'=au+b,\ v'=cv+d,\ a,c>0\}$$

and $(\mu',\nu')=(a\mu+b,\ c\nu+d)$

Then $A(\mathcal{R}',(\mu',\nu'))=(a\alpha+b,\ c\beta+d)$

6. (Symmetric) If $\mathcal{R}$ is symmetric, i.e., $(u,v)\in\mathcal{R}\Rightarrow(v,u)\in\mathcal{R}$, and $\mu=\nu$, then $\alpha=\beta$



Example:

1. Dating game: $(A,B)=\begin{pmatrix}(4,2) & (0,0)\\ (0,0) & (1,3)\end{pmatrix}$

$A=\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}$ $\begin{matrix}4\\-1\end{matrix} \times \begin{matrix}0.25\\0.75\end{matrix}$ $\qquad B=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$

$\begin{matrix}4 & & -1\\ & \times & \\ 0.25 & & 0.75\end{matrix}$

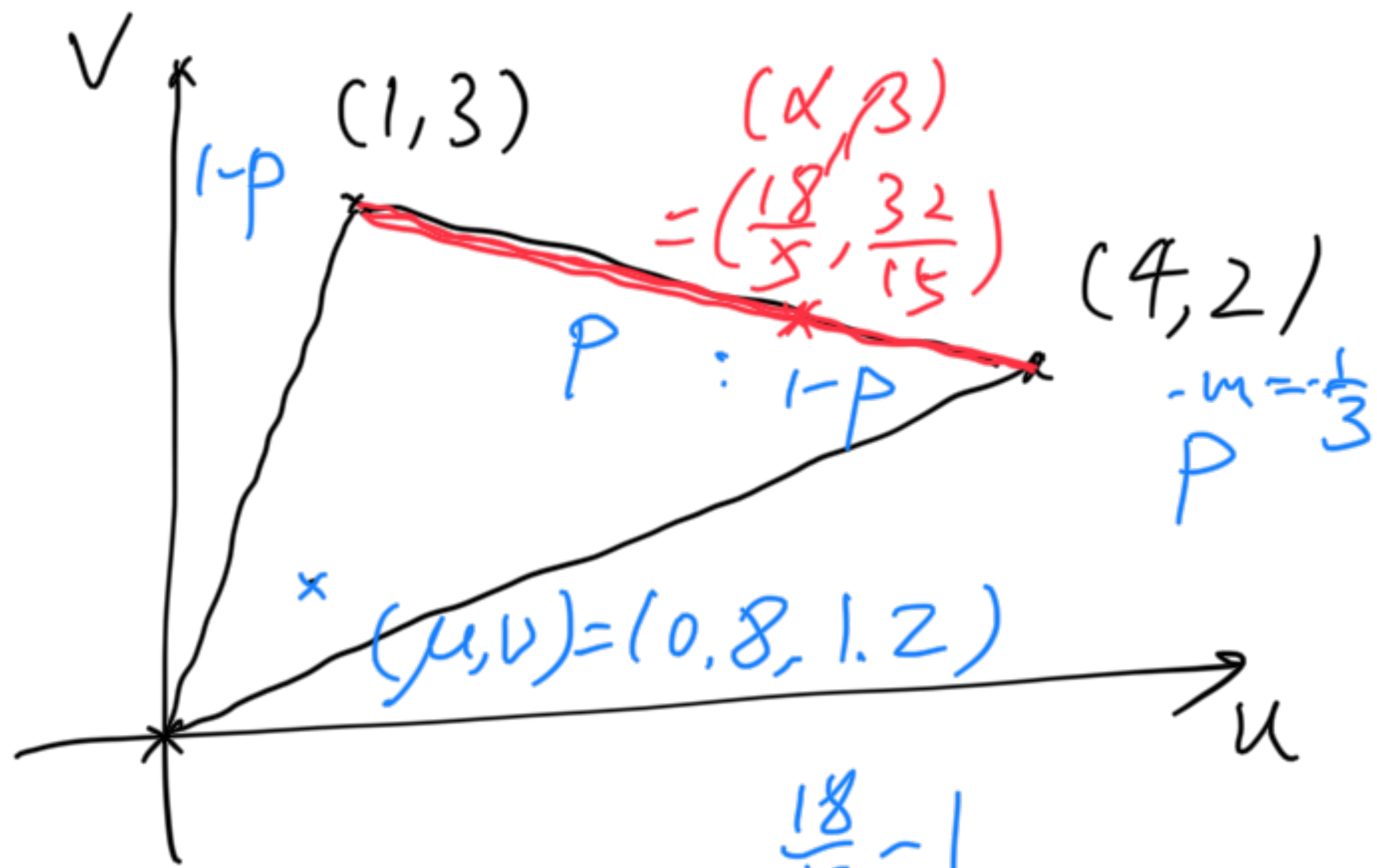
$\mu = v(A) = 0.8 \qquad \nu = v(B^T) = 1.2$

$(\mu,\nu) = (0.8, 1.2)$

$v-3 = -\frac{1}{3}(u-1)$

$v = -\frac{1}{3}u + \frac{10}{3}$

② $\left(\frac{18}{5}, \frac{32}{15}\right) = p(4,2) + (1-p)(1,3)$



$g(u,v) = (u-\mu)(v-\nu)$

$= (u-0.8)(v-1.2)$

$= \left(u-\frac{4}{5}\right)\left(-\frac{1}{3}u+\frac{10}{3}-\frac{6}{5}\right)$

① $p = \dfrac{\frac{18}{5}-1}{4-1}$

$= \dfrac{3-\frac{32}{15}}{\ldots} = \frac{13}{\ldots}$

$$= -\frac{1}{3}u^2 + \frac{12}{5}u - \frac{128}{75}$$

(α, β) is the point on the bargaining where $g(u,v) = (u-\mu)(v-\nu)$ attains its maximum.

$$\alpha = \frac{\frac{12}{5}}{2 \times \frac{1}{3}} = \frac{18}{5}, \quad \beta = -\frac{1}{3} \times \frac{18}{5} + \frac{10}{3} = \frac{32}{15}$$

$\therefore$ The arbitration pair is $(\alpha, \beta) = \left(\frac{18}{5}, \frac{32}{15}\right)$

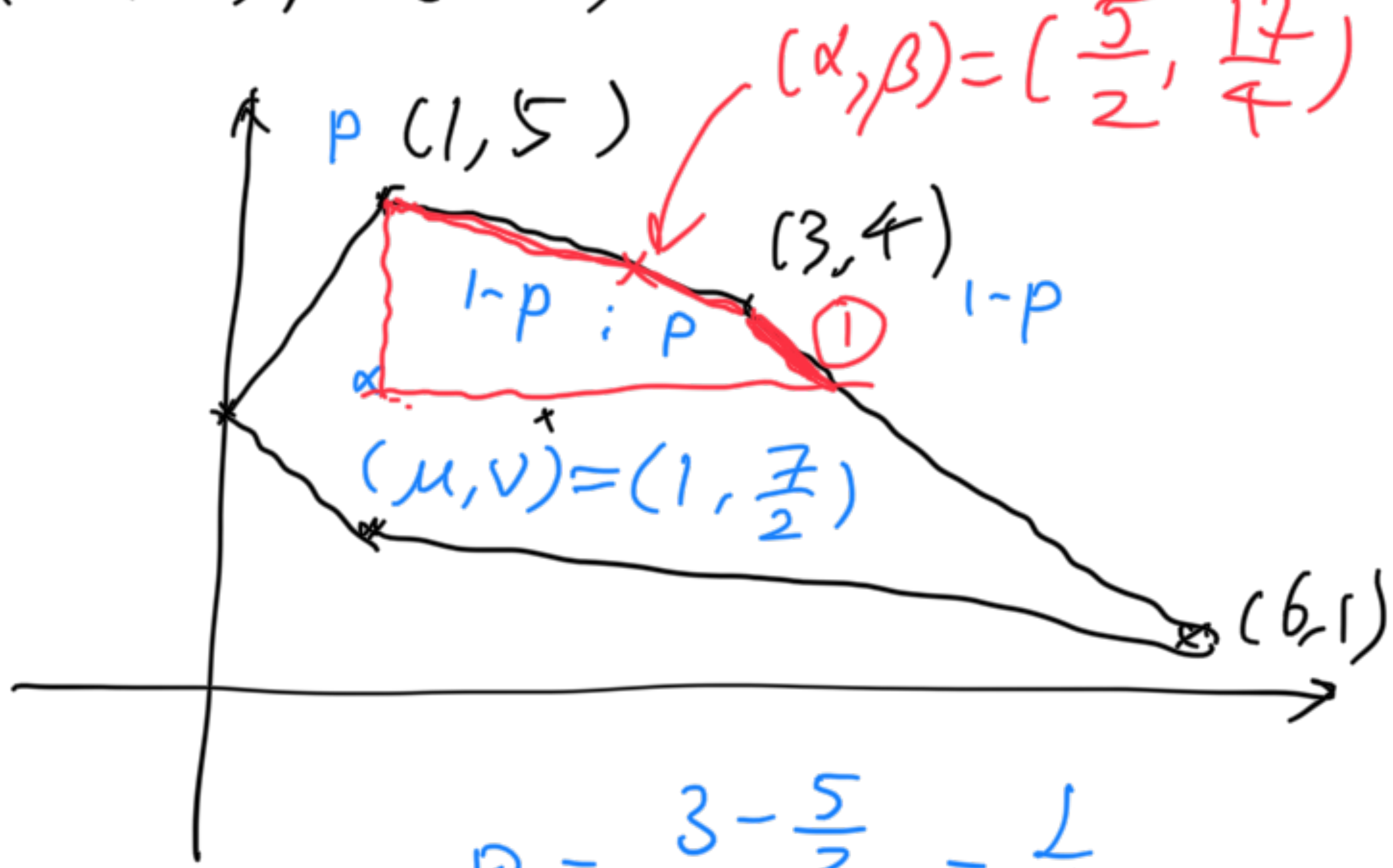
The joint strategy is

$$P = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \quad \text{where} \quad p = \frac{\frac{18}{5} - 1}{4-1} = \frac{13}{15}$$

$$= \begin{pmatrix} \frac{13}{15} & 0 \\ 0 & \frac{2}{15} \end{pmatrix}$$

2. $$(A, B) = \begin{pmatrix} (2,3) & (1,5) & (6,1) \\ (3,4) & (1,2) & (0,3) \end{pmatrix}$$

$$(\mu,\nu) = \left(1, \frac{7}{2}\right) = (\nu(A), \nu(B^T))$$



$$p = \frac{3-\frac{5}{2}}{3-1} = \frac{1}{4}$$

① $v-4=-(u-3)$

$$v = -u+7$$

$$g(u,v) = (u-1)\left(v-\frac{7}{2}\right) = (u-1)\left(-u+7-\frac{7}{2}\right) = (u-1)\left(-u+\frac{7}{2}\right) = -u^2+\frac{9}{2}u-\frac{7}{2}$$

maximum when $u = \frac{+\frac{9}{2}}{2} = +\frac{9}{4} = 2.25 < 3$

$\boxed{\left(\frac{9}{4}, \frac{19}{4}\right)}$ lies outside $\mathcal{R}$

$$\begin{cases} v-4=-(u-3) \\ v-\frac{7}{2}=(u-1) \end{cases} \Rightarrow (u,v)=\left(\frac{9}{4},\frac{19}{4}\right)$$

② $v-5=-\frac{1}{2}(u-1)$

$$v=-\frac{1}{2}u+\frac{11}{2}$$

$$g(u,v)=(u-1)\left(v-\frac{7}{2}\right)=(u-1)\left(-\frac{1}{2}u+2\right)$$

$$= -\frac{1}{2}u^2+\frac{5}{2}u-2$$

maximum when $u = \frac{5}{2}$

$\left(\frac{5}{2}, \frac{17}{4}\right)$ lies on the line segment.

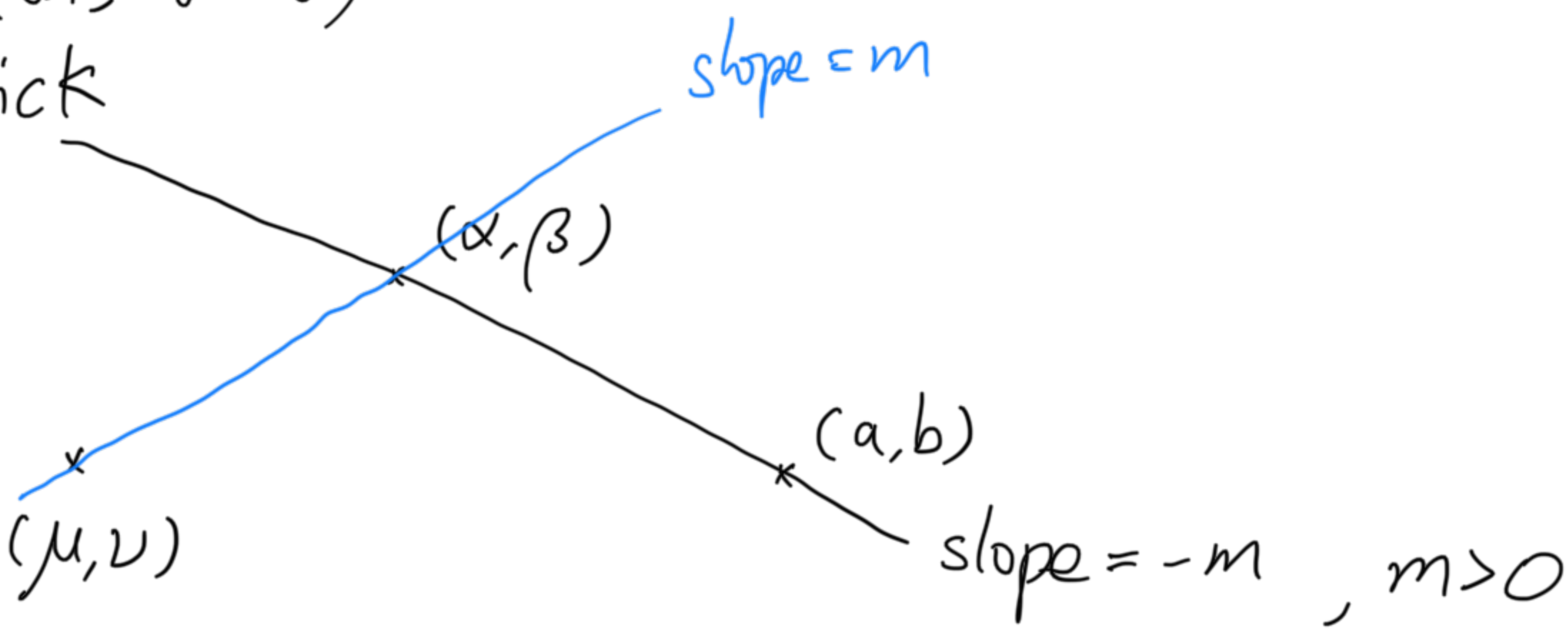
$\therefore$ The arbitration pair is $(\alpha, \beta) = \left(\frac{5}{2}, \frac{17}{4}\right)$

The joint strategy:

$$P = \begin{pmatrix} 0 & p & 0 \\ 1-p & 0 & 0 \end{pmatrix} \quad \text{where } p = \frac{3-\frac{5}{2}}{3-1} = \frac{1}{4}$$

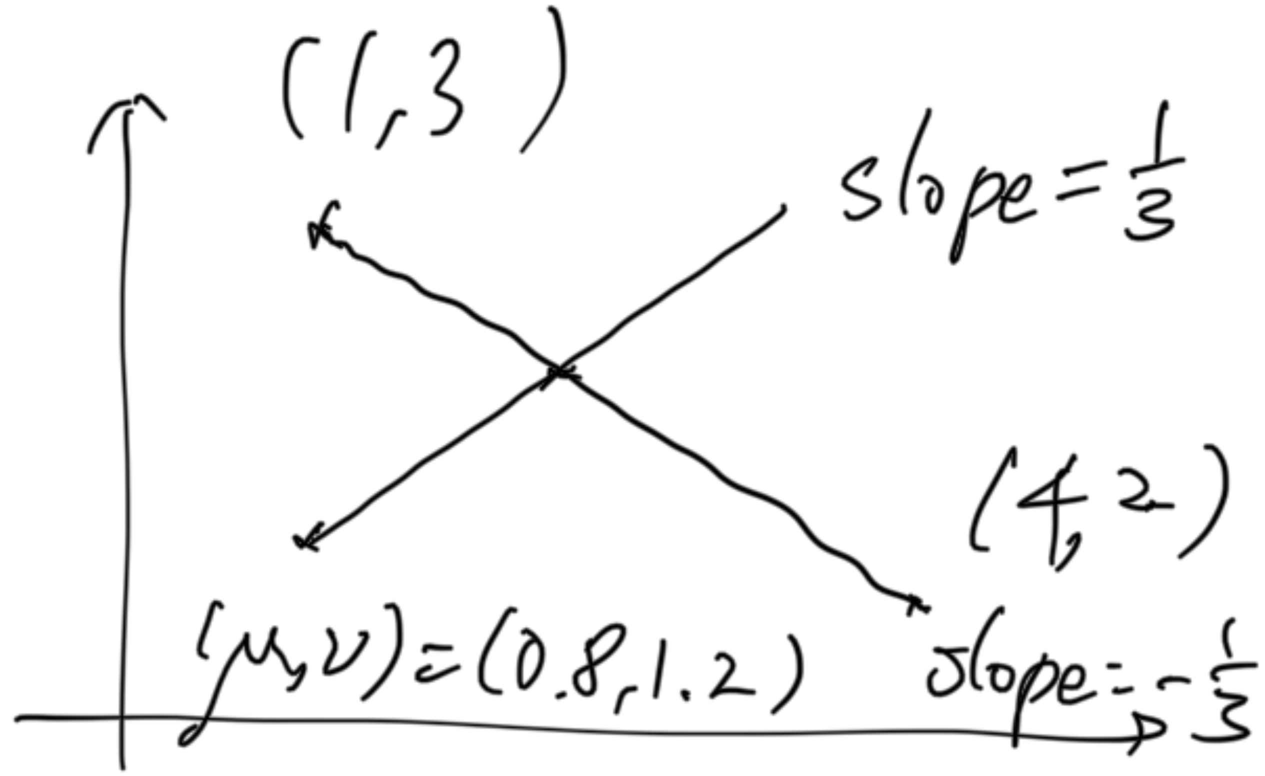
$$= \begin{pmatrix} 0 & 0.25 & 0 \\ 0.75 & 0 & 0 \end{pmatrix}$$

Little trick



$\beta - b = -m(\alpha - a)$

$\left(\beta - \nu = m(\alpha - \mu) \right)$

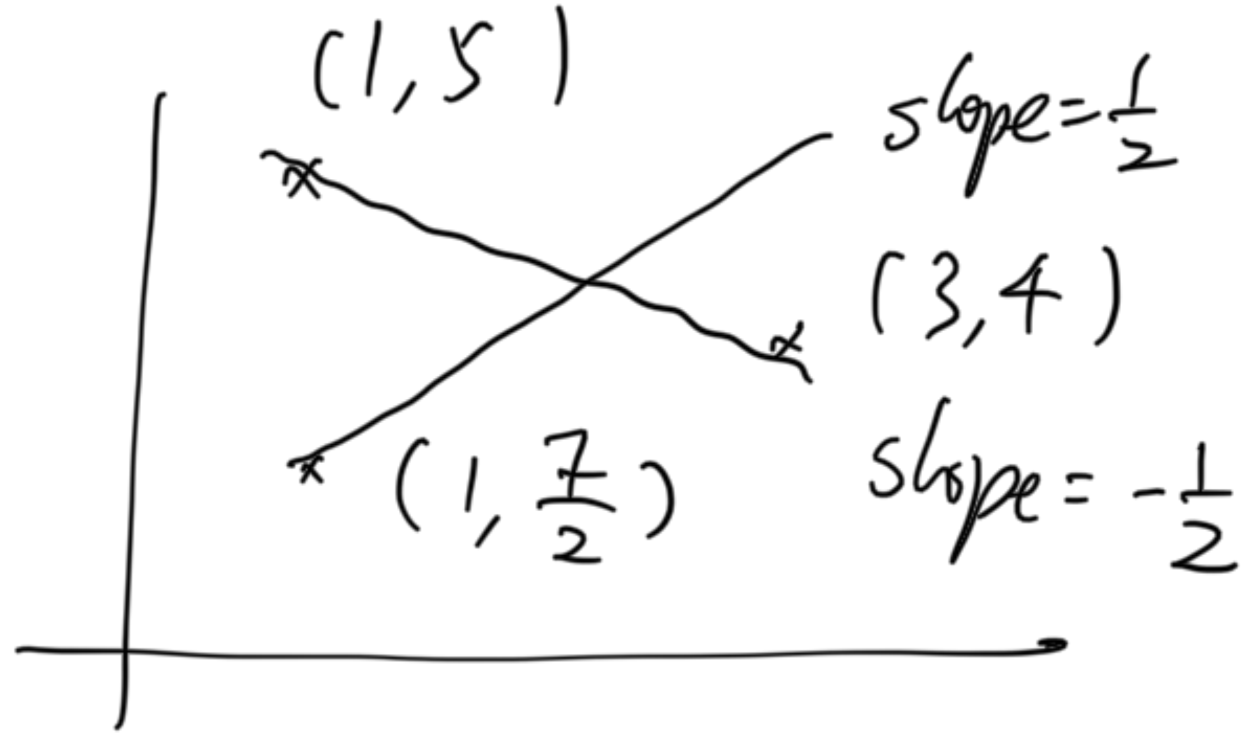


In dating game:

$$\begin{cases} v - 3 = -\frac{1}{3}(u-1) \\ v - 1.2 = \frac{1}{3}(u - 0.8) \end{cases}$$

$$\Rightarrow (u,v) = \left(\frac{18}{5}, \frac{32}{15}\right)$$

In the second example



$$\begin{cases} v - 5 = -\frac{1}{2}(u-1) \\ v - \frac{7}{2} = \frac{1}{2}(u-1) \end{cases}$$

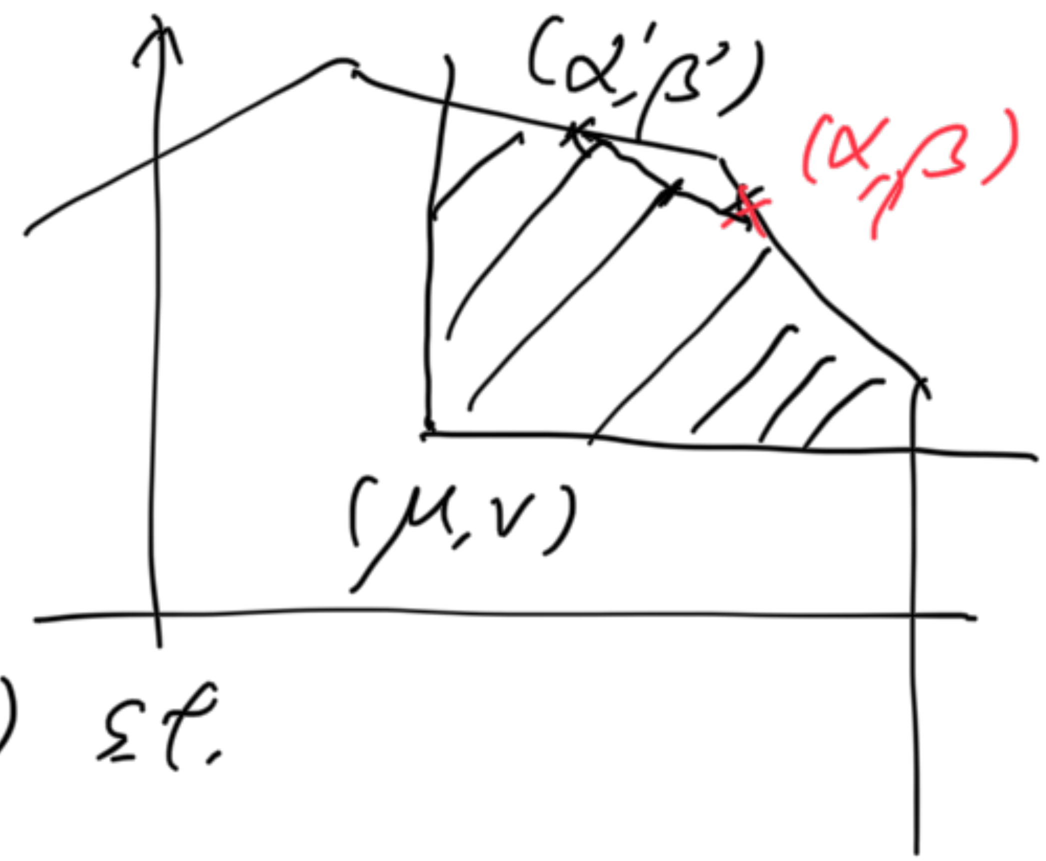
$$\Rightarrow (u,v) = \left(\frac{5}{2}, \frac{17}{4}\right)$$

Thm (Nash) There <u>exists</u> a <u>unique</u> arbitration $A(R, (\mu, \nu))$ satisfies the Nash's axioms.

Lemma: Define

$$g(u,v) = (u-\mu)(v-\nu) \quad \text{for } (u,v) \in \mathcal{R} \text{ with } (u,v) \geq (\mu,\nu)$$

Then $\exists$ unique (α,β) s.t.



$$g(\alpha,\beta) = \max_{\substack{(u,v)\in\mathcal{R} \\ (u,v)\geq(\mu,\nu)}} g(u,v)$$

Proof. Suppose $\exists\ (\alpha',\beta') \neq (\alpha,\beta)$ s.t.

$$g(\alpha',\beta') = g(\alpha,\beta)$$

Then either $\alpha' > \alpha$ and $\beta' < \beta$ or $\alpha' < \alpha$ and $\beta' > \beta$

$$g\left(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2}\right)$$

$$= \left(\frac{\alpha+\alpha'}{2} - \mu\right)\left(\frac{\beta+\beta'}{2} - \nu\right)$$

$$= \frac{1}{4}\Big(\big((\alpha-\mu)+(\alpha'-\mu)\big)\big((\beta-\nu)+(\beta'-\nu)\big)\Big)$$

$$= \frac{1}{4}\Big[(\alpha-\mu)(\beta-\nu) + (\alpha-\mu)(\beta'-\nu) + (\alpha'-\mu)(\beta-\nu) + (\alpha'-\mu)(\beta'-\nu)\Big)$$

$$= \frac{1}{4}\Big[(\alpha-\mu)(\beta-\nu) + (\alpha-\mu)(\beta-\nu+\beta'-\beta) + (\alpha'-\mu)(\beta'-\nu+\beta-\beta') + (\alpha'-\mu)(\beta'-\nu)\Big]$$

$$= \frac{1}{4}\Big[g(\alpha,\beta) + \underline{(\alpha-\mu)(\beta-\nu)} + (\alpha-\mu)(\beta'-\beta) + \underline{(\alpha'-\mu)(\beta'-\nu)} + (\alpha'-\mu)(\beta-\beta') + g(\alpha',\beta')\Big]$$

$$= \frac{1}{4}\Big(2g(\alpha,\beta) + 2g(\alpha',\beta') + (\beta'-\beta)\big((\alpha-\mu)-(\alpha'-\mu)\big)\Big]$$

$$= \frac{1}{4}\Big(2g(\alpha,\beta) + 2g(\alpha',\beta') + (\beta'-\beta)(\alpha-\alpha')\Big]$$

$$> g(\alpha,\beta)$$

Contradicts $\left(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2}\right) \in R$

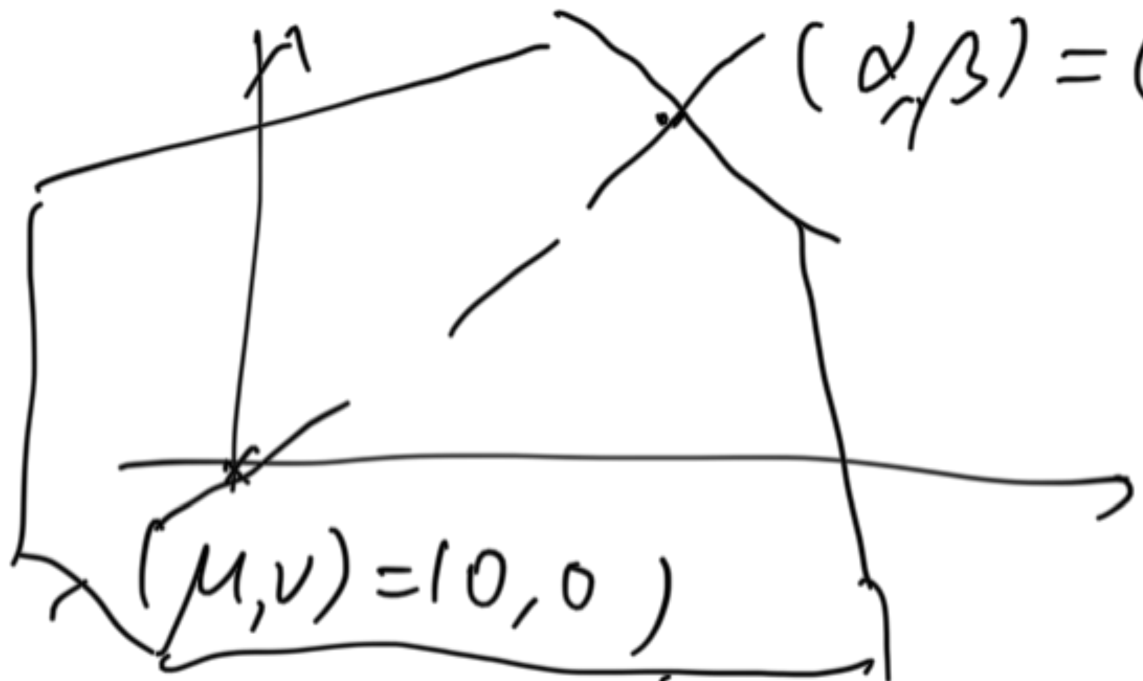
and g attains its maximum at (α,β).

Define $A(R,(\mu,\nu)) = (\alpha,\beta)$ s.t.

Define $A(R,(\mu,\nu)) = (\alpha,\beta)$ s.t.

$$g(\alpha,\beta) = \max_{\substack{(u,v)\in R \\ (u,v)\geq(\mu,\nu)}} g(u,v)$$

To prove uniqueness, suppose A' is another arbitration function. By taking a linear transformation, we may assume $(\mu,\nu)=(0,0)$ and $(\alpha,\beta)=(1,1)$



Use 4, 6 $\Rightarrow A'(R,(0,0)) = (1,1)$